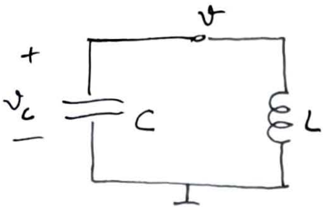
SECOND-ORDER CIRCUITS / SYSTEMS

Undriven L-C circuit :



Undriven response is also known as Zero Input Response (ZIR) of circui

$$C \frac{dv(t)}{dt} + \frac{1}{L} \int_{-\infty}^{t} v(t)\,dt = 0 \quad \Rightarrow \quad \frac{d^2 v(t)}{dt^2} + \frac{1}{LC} v(t) = 0$$

we set Ae^{st} or using Laplace

$$A s^2 e^{st} + A \frac{1}{LC} e^{st} = 0 \quad \Rightarrow \quad A\underbrace{(s^2 + \frac{1}{LC})}_{0} e^{st} = 0$$

$$s_1 = +j\omega_o \qquad \text{where} \quad \omega_o = \frac{1}{\sqrt{LC}}$$
$$s_2 = -j\omega_o$$

Since $j = \sqrt{-1}$. The solution for v is linear combination of $e^{s_1 t}$ and $e^{s_2 t}$.

so $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{j\omega_o t} + A_2 e^{-j\omega_o t}$

$$\boxed{e^{jx} = \cos x + j \sin x}$$

Therefore we can write : $v(t) = K_1 \cos \omega_o t + K_2 \sin \omega_o t$

$e^{j\omega_o t} + e^{-j\omega_o t} = 2\cos \omega_o t$

we calculate K_1 and K_2 using $v(0)$, $\frac{dv}{dt}(0)$ initial values.

$e^{j\omega_o t} - e^{-j\omega_o t} = 2j \sin \omega_o t$

$$v(0) = K_1 \cos \phi t + K_2 \sin \phi t = K_1 \qquad \frac{dv}{dt}(0) = \omega_o K_2 \quad \Rightarrow \quad K_2 = \frac{1}{\omega_o} \frac{dv}{dt}(0)$$

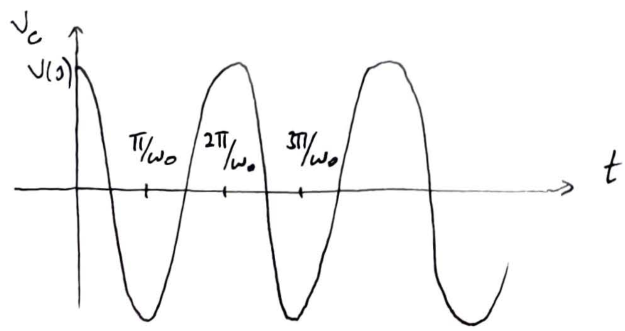
So $$v(t) = v(0) \cdot \cos \omega_o t + \frac{1}{\omega_o} \frac{dv}{dt}(0) \sin(\omega_o t)$$

Or using Laplace Transform :

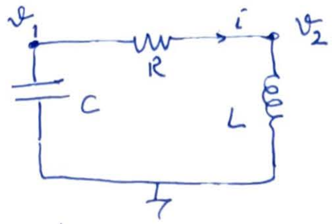
$$\frac{d^2}{dt^2} v(t) + \frac{1}{LC} v(t) = 0 \quad \Rightarrow \quad s^2 V(s) - s v(0) - \dot{v}(0) + \frac{1}{LC} V(s) = 0$$

$$V(s) = \frac{s v(0) + \dot{v}(0)}{s^2 + \frac{1}{LC}} = \frac{s v(0) + \dot{v}(0)}{s^2 + \omega_o^2} = A \frac{s}{s^2 + \omega_o^2} + B \frac{\omega_o}{s^2 + \omega_o^2}$$

$$\Rightarrow \quad s v(0) = A s \qquad B \omega_o = \dot{v}(0) \qquad \Rightarrow \quad v(t) = v(0) \cos \omega_o t + \frac{1}{\omega_o} \dot{v}(0) \sin \omega_o t$$
$$A = v(0) \qquad B = \frac{1}{\omega_o} \dot{v}(0)$$



Undriven series RLC :



KCL @ Node 1 : $C \dfrac{dv_1}{dt} + \dfrac{v_1 - v_2}{R} = 0$

KCL @ Node 2 : $\dfrac{v_2 - v_1}{R} + \dfrac{1}{L} \displaystyle\int_{-\infty}^{t} v_2 \, dt = 0$

or $\quad Ri + L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int i \, dt = 0$

$\Rightarrow \quad \ddot{i} + \dfrac{R}{L}\dot{i} + \dfrac{1}{LC} i = 0 \qquad \dfrac{d^2}{dt^2} i(t) + \dfrac{R}{L}\dfrac{d\,i(t)}{dt} + \dfrac{1}{LC} i(t) = 0$

$i = Ae^{st}$ is the solution. $\Rightarrow \quad A\underbrace{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}_{\text{characteristic equation}} = 0$

we use this general form for characteristic equation of 2^{nd} order circuits :

$$s^2 + 2\alpha s + \omega_0^2 = 0 \quad \text{where} \quad \alpha \equiv \frac{R}{2L} \;,\quad \omega_0 \equiv \frac{1}{\sqrt{LC}}$$

so the roots : $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$
$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$ $\quad$ Decaying and oscillatory shape !

Therefore the solution is linear combination of $e^{s_1 t}$ and $e^{s_2 t}$:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

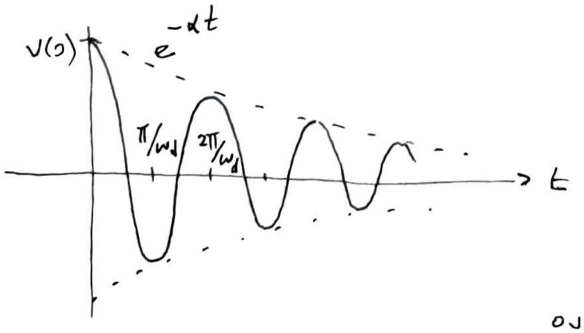
we have three cases :

$\alpha < \omega_0$	$\Rightarrow$	under-damped dynamics
$\alpha = \omega_0$	$\Rightarrow$	critically-damped dynamics
$\alpha > \omega_0$	$\Rightarrow$	over-damped dynamics

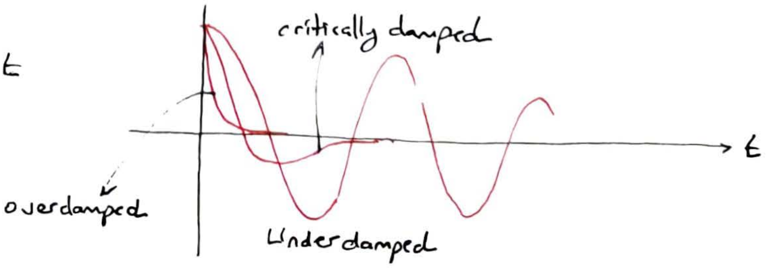
Underdamped dynamics :

$\alpha < \omega_0$ or $R/2 < \sqrt{L/C}$ $\quad$ if $\omega_d \equiv \sqrt{\omega_0^2 - \alpha^2}$ then $\quad s_1 = -\alpha + j\omega_d$
$s_2 = -\alpha - j\omega_d$

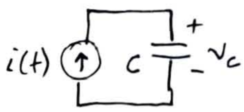
and $e^{j\omega_d t} = \cos\omega_d t + j\sin\omega_d t$



if $R = 0$, $\alpha = 0$ then signal oscillates forever!



* STATE, STATE VARIABLES, COMPUTER SIMULATIONS



$$q(t) = \int_{-\infty}^{t} i(t)\,dt$$

This is a memory element. $q(t)$ depends all the past values. But if we know q at time t_1, then:

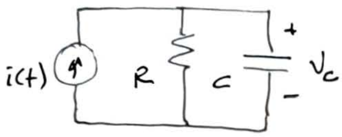
$$q(t_2) = \int_{-\infty}^{t_1} i(t)\,dt + \int_{t_1}^{t_2} i(t)\,dt = q(t_1) + \int_{t_1}^{t_2} i(t)\,dt$$

q is a state variable. Since $q = C \cdot v$, capacitor voltage is a state variable.
Similarly inductor current is a state variable.
State equations are like this:

$$\frac{d}{dt}(\text{state variable}) = f(\text{state variable, input variable})$$ if f is linear,

$$\frac{d}{dt}(\text{state variable}) = K_1 \cdot (\text{state var. present value}) + K_2 \cdot (\text{input variable})$$

Consider



$$i(t) = \frac{v_c}{R} + C\frac{dv_c}{dt} \Rightarrow \frac{dv_c}{dt} = -\frac{v_c}{RC} + \frac{i(t)}{C}$$

present value

input value

v_c is the state variable

Computer analysis:

Recall Euler's method:

$$\frac{dv_c(t)}{dt} \approx \frac{v_c(t+\Delta t) - v_c(t)}{\Delta t} \Rightarrow v_c(t_0+\Delta t) = v_c(t_0) - \frac{v_c(t_0)}{RC}\Delta t + \frac{i(t_0)}{C}\cdot\Delta t$$

Python code: 1

rc.py

solves $\dot{v}_c = -\frac{1}{RC}v_c + \frac{i}{C}$! (Above circuit)

State Equations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Python code: 2

Undriven LC circuit

lc.py

State equations are derived:

$$v = L\dot{i}, \quad i = C\dot{v} \Rightarrow \begin{aligned} i(t+\Delta t) &= i(t) + \frac{1}{L}v(t)\,\Delta t \\ v(t+\Delta t) &= v(t) - \frac{1}{C}i(t+\Delta t)\,\Delta t \end{aligned}$$